Torus Knots and the Rational DAHA

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1 Part I

Definition 1.1. *Let L be a representation of S_n. Define the Frobenius character map* $\text{ch} : \text{Rep } S_n \to \Lambda_n$ *(where Λ_n =symmetric polynomials of degree n) to be*

$$\text{ch}(L) = \frac{1}{n!}\sum_{\sigma \in S_n} \text{Tr}_L(\sigma) p_1^{k_1(\sigma)} \dots p_r^{k_r(\sigma)}$$

where p_i are power sums, $k_i(\sigma)$ is the number of cycles of length i in σ.

Remark. $\text{ch}(S^\lambda) = s_\lambda$. Note $[S^\lambda]_{\lambda \vdash n}$ forms a basis for $K_0(\text{Rep } S_n)$ and in fact

$$\text{ch} : K_0\left(\bigoplus_{n\geq 0} \text{Rep } S_n\right) \xrightarrow{\sim} \Lambda(= \text{ symmetric polynomials in } \infty \text{ many variables})$$

is an isomorphism of Hopf Algebras. (Representations of the Symmetric Group is a categorification of symmetric functions.)

Lemma 1.2. *The reflection(geometric) representation $\mathfrak{h}$ of S_n is isomorphic to $\mathbb{C}^n/\mathbb{C} \cdot x_1 + \ldots + x_n$ where $\mathbb{C}^n$ is the defining representation of S_n.*

Lemma 1.3. *Let $T : V \to V$ be a linear operator and let $V = V_1 \oplus \ldots V_k$ where each V_i is $T-$invariant. Then*

$$\text{char}_T(q) = \text{char}_{T|_{V_i}}(q) \dots \text{char}_{T|_{V_n}}(q)$$

Proof. $qI - T$ will be a block matrix. ■

Proposition 1.4. *For $\sigma \in S_n$ acting in the reflecting representation $\mathfrak{h}$*

$$\det_{\mathfrak{h}}(I - q\sigma) = \frac{1}{1-q}\prod_i (1-q^i)^{k_i(\sigma)} \tag{1}$$

Proof. It is easy to see that for $A : V \to V$ where V is n dimensional,

$$det(I - qA) = (-q)^n \text{char}_A(q^{-1}) \tag{2}$$

From Lemma 1.2 we have that $\mathbb{C}^n = \mathbb{C} \oplus \mathfrak{h}$ as representations and so by Lemma 1.3

$$\det_{\mathfrak{h}}(I - q\sigma) = \frac{\det_{\mathbb{C}^n}(I - q\sigma)}{\det_{\text{triv}}(I - q\sigma)} = \frac{\det_{\mathbb{C}^n}(I - q\sigma)}{1-q}$$

As the characteristic polynomial is conjugation invariant in $\text{GL}(\mathbb{C}^n)$, and conjugating by permutation matrices corresponds to conjugation in S_n so we see that the LHS above only depends on the cycle type of σ. For each cycle c in σ of length i, notice there is a σ invariant subspace V_c of $\mathbb{C}^n$ of dimension i.

For example, if $\sigma = (1234)(56)$, then $V_{(1234)} = \oplus_{i=1}^4 \mathbb{C}x_1$ and $V_{(56)} = \mathbb{C}x_5 \oplus \mathbb{C}x_6$ are our two σ invariant subspaces. It is clear these only depend on the length i and that if $\sigma = c_1 \dots c_m$ where c_i are cycles,

$$\mathbb{C}^n = V_{c_1} \oplus \dots \oplus V_{c_m}$$

Therefore by Lemma 1.3 we see that

$$\det_{\mathbb{C}^n}(I - q\sigma) = \prod_i \det_{\mathbb{C}^i}(I - q(12\cdots i))^{k_i(\sigma)}$$

where $T_i = (12\cdots i)$ acts on $\mathbb{C}^i$ by permutation of basis vectors. It's clear that $\mathbb{C}^i$ is a $T-$cyclic vector space, i.e. $\left\{T^j(x_1)\right\}_{j\geq 0} = \mathbb{C}^i$. As a result,

$$\text{char}_{T_i}(q) = (-1)^i \text{min}_{T_i}(q)$$

and so $\deg \text{min}_{T_i}(q) = i$. Because $T_i^i - I = 0$ it follows that $\text{min}_{T_i}(q) = q^i - 1$. Thus

$$\det_{\mathbb{C}^i}(I - q(12\cdots i)) = (-q)^i \text{char}_{T_i}(q) = q^i\left(\frac{1}{q^i} - 1\right) = 1 - q^i$$

■

Recall $L_{m/n} = \bigoplus_i (L_{m/n})_i$ where each $(L_{m/n})_i$ is a representation of S_n.

Proposition 1
Let $F_{m/n}(q, p_i) := \text{gch}(L_{m/n}) := \sum_i \text{ch}((L_{m/n})_i)q^i$. Fixing m, we claim

$$F_m(q, p_i) := \sum_{n=0}^{\infty} F_{m/n}(q, p_i)z^n = \frac{1}{[m]_q}\prod_{j=0}^{m-1}\prod_{k=1}^{\infty}\frac{1}{1 - q^{j+\frac{1-m}{2}}zx_k}$$

where $[m]_q = \dfrac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}}$.

Proof. Let $\delta_{m,n} = \dfrac{(m-1)(n-1)}{2}$. Using what Sam wrote,

$$\sum_i \text{Tr}(\sigma, (L_{m/n})_i)q^i = q^{-\delta_{m,n}}\frac{\det_{\mathfrak{h}}(1 - q^m\sigma)}{\det_{\mathfrak{h}}(1 - q\sigma)} \overset{\text{Eq. (1)}}{=\!=} q^{-\delta_{m,n}}\frac{1 - q}{1 - q^m}\prod_i\left(\frac{1 - q^{mi}}{1 - q^i}\right)^{k_i(\sigma)}$$

Thus

$$\begin{aligned} F_{m/n}(q, p_i) &= \sum_i \frac{1}{n!}\sum_{\sigma\in S_n} \text{Tr}(\sigma, (L_{m/n})_i)p_1^{k_1(\sigma)}\dots p_r^{k_r(\sigma)}q^i \\ &= \frac{1}{n!}\sum_{\sigma\in S_n}\frac{q^{\frac{n(1-m)}{2}}}{[m]_q}\prod_i\left(\frac{1 - q^{mi}}{1 - q^i}p_i\right)^{k_i(\sigma)} \end{aligned}$$

as $q^{-\delta_{m,n}}\dfrac{1-q}{1-q^m} = \dfrac{q^{\frac{n(1-m)}{2}}}{[m]_q}$. Thus,

$$\begin{aligned}
F_m(q,p_i) &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\sigma\in S_n}\frac{q^{\frac{n(1-m)}{2}}}{[m]_q}\prod_i\left(\frac{1-q^{mi}}{1-q^i}p_i\right)^{k_i(\sigma)} z^n && (3)\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\lambda\vdash n}\frac{n!}{\prod_i i^{k_i(\lambda)}k_i(\lambda)!}\frac{q^{\frac{n(1-m)}{2}}}{[m]_q}\prod_i\left(\frac{1-q^{mi}}{1-q^i}p_i\right)^{k_i(\lambda)} z^n && (4)\\
&= \frac{1}{[m]_q}\sum_{n=0}^{\infty}\sum_{\lambda\vdash n}\prod_i\frac{1}{k_i(\lambda)!}\left(\frac{(1-q^{mi})p_i q^{\frac{i(1-m)}{2}}z^i}{(1-q^i)i}\right)^{k_i(\lambda)} && (5)
\end{aligned}$$

where going from (3) − (4), the cycle type of an element $\sigma\in S_n$ is the same as a partition λ of n. The number of permutations in S_n with cycle type λ is precisely the size of the conjugacy class in S_n so we can then reindex over partitions of n. Going from (4) − (5) we use $\sum_i ik_i(\lambda) = n$. Now note

$$\prod_i\frac{1}{1-z^i} = \sum_{n=0}^{\infty}\left(\sum_{\lambda\vdash n}1\right)z^n = \sum_{n=0}^{\infty}\sum_{\lambda\vdash n}\prod_i (z^i)^{k_i(\lambda)}$$

$$\implies \prod_i\frac{1}{1-f(q,i)z^i} = \sum_{n=0}^{\infty}\sum_{\lambda\vdash n}\prod_i (f(q,i)z^i)^{k_i(\lambda)}$$

Moving over to exponential generating functions it follows that

$$\prod_i \exp\left(f(q,i)z^i\right) = \sum_{n=0}^{\infty}\sum_{\lambda\vdash n}\prod_i\frac{1}{k_i(\lambda)!}(f(q,i)z^i)^{k_i(\lambda)}$$

and thus

$$\begin{aligned}
F_m(q,p_i) &= \frac{1}{[m]_q}\exp\left(\sum_{i=1}^{\infty}\frac{(1-q^{mi})p_i q^{\frac{i(1-m)}{2}}z^i}{(1-q^i)i}\right) && (6)\\
&= \frac{1}{[m]_q}\prod_{j=0}^{m-1}\exp\left(\sum_{i=1}^{\infty}\frac{q^{ij}p_i q^{\frac{i(1-m)}{2}}z^i}{i}\right) && (7)\\
&= \frac{1}{[m]_q}\prod_{j=0}^{m-1}\exp\left(\sum_{i=1}^{\infty}\frac{(q^{j+\frac{(1-m)}{2}}z)^i}{i}p_i\right) && (8)\\
&= \frac{1}{[m]_q}\prod_{j=0}^{m-1}\prod_{k=1}^{\infty}\exp\left(\log\left(\frac{1}{1-q^{j+\frac{(1-m)}{2}}zx_k}\right)\right) && (9)
\end{aligned}$$

■

2 Part II

- Rewrite Proposition 1.
- $\mathrm{ch}(L)$ is really the same datum as $\{\chi_L(g)\}_{g\in S_n}$. For example, the cycle type for the identity permutation is (1^n), so $\langle p_1^n\rangle\,\mathrm{ch}L = \dfrac{\dim L}{n!} \implies \dim L = n!\,\langle \mathrm{ch}L, p_1^n\rangle$.

The advantage of using $\mathrm{ch}(L)$ is that it's a generating function/formal. Notice characters for S_n is a function while characters for $\mathrm{GL}(m) =$ is a generating function.

$$\begin{array}{ccc} K_0\left(\bigoplus_{n\geq 0} \text{Rep } S_n\right) & & K_0\left(\text{Rep}^{poly}\ \mathrm{GL}(m)\right) \\ & \underset{\text{ch}}{\overset{\cong}{\searrow}} \quad \sim \text{ for } m>>0 \quad \underset{\mathrm{Tr}(\mathrm{diag}(x_1,\ldots,x_m),-)}{\swarrow} & \\ & \text{Sym} & \end{array}$$

Theorem 2

As a graded S_n-rep, the representation $L_{m/n}$ decomposes as

$$L_{m/n} = \frac{1}{[m]_q} \bigoplus_{\lambda\vdash n} s_\lambda(q^{\frac{1-m}{2}}, q^{\frac{3-m}{2}}, \ldots, q^{\frac{m-1}{2}}) S^\lambda \tag{10}$$

Proof. Since ch is an isomorphism it suffices to show this at the level of graded Frobenius characters. Now, using the Cauchy identity

$$\prod_{k,j} \frac{1}{1-x_k y_j} = \sum_\lambda s_\lambda(x_1,\ldots)s_\lambda(y_1,\ldots)$$

with

$$y_j = \begin{cases} zq^{j+\frac{1-m}{2}} & 0\leq j<m \\ 0 & j\geq m \end{cases}$$

we see that

$$\begin{aligned} \mathrm{gch}(L_{m/n}) = \langle z^n\rangle\, F_m(q,p_i) &\overset{\text{Proposition 1}}{=\!=} \langle z^n\rangle \frac{1}{[m]_q} \prod_{j=0}^{m-1}\prod_{k=1}^{\infty} \frac{1}{1-q^{j+\frac{1-m}{2}}zx_k} \\ &\overset{\text{Cauchy}}{=\!=} \langle z^n\rangle \frac{1}{[m]_q} \sum_\lambda s_\lambda(x_1,\ldots)s_\lambda(zq^{\frac{1-m}{2}}, zq^{\frac{3-m}{2}}, \ldots, zq^{\frac{m-1}{2}}) \\ &= \langle z^n\rangle \frac{1}{[m]_q} \sum_n \sum_{\lambda\vdash n} z^n s_\lambda(q^{\frac{1-m}{2}}, q^{\frac{3-m}{2}}, \ldots, q^{\frac{m-1}{2}}) s_\lambda(x_1,\ldots) \\ &= \frac{1}{[m]_q} \sum_{\lambda\vdash n} s_\lambda(q^{\frac{1-m}{2}}, q^{\frac{3-m}{2}}, \ldots, q^{\frac{m-1}{2}}) s_\lambda(x_1,\ldots) \end{aligned}$$

■

Proposition 2.1 (Hook-Content Formula)**.**

$$s_\lambda(q^{\frac{1-m}{2}}, q^{\frac{3-m}{2}}, \ldots, q^{\frac{m-1}{2}}) = \prod_{(i,j)\in\lambda} \frac{[m+i-j]_q}{[h_\lambda(i,j)]_q}$$

where $[m]_q = \dfrac{q^{m/2}-q^{-m/2}}{q^{1/2}-q^{-1/2}}$ *and* $h_\lambda(i,j)$ *=hook length of box* (i,j)*.*

Example.

$$L_{3/2} = (q+q^{-1})S^{\square\square} \bigoplus S^{\substack{\square\\\square}}$$

3 HOMFLY polynomial of Torus knots

Definition 3.1. *Recall that the HOMFLY polynomial $P = P_{q-q^{-1}}(a, q)$ of a link L is defined to be*

$$aP(L_+) - a^{-1}P(L_-) = (q - q^{-1})P(L_0)$$

and $P(\text{unknot}) = 1$.

Example. For the trefoil ($T(2,3)$) one can compute

$$P(T(2,3)) = a^2(q^2 + q^{-2} - a^2)$$

Definition 3.2. *$H_q(n)$ is the quotient of $\mathbb{Z}[q^{\pm 1}][B_n]$ by the relation*

$$\sigma_i^2 = (q-1)\sigma_i + q$$

where $\{\sigma_i\}_{1\le i\le n-1}$ be the usual set of generators for B_n. Let $g_i := [\sigma_i] \in H_q(n)$.

Warning. q is always generic!

Definition 3.3. *The Jones-Ocneanu trace* $\operatorname{tr} : \bigcup_{n\ge 1} H_q(n) \to \mathbb{Z}[q^{\pm 1}][z]$ *is the unique linear map s.t.*

(1) $\operatorname{tr}(ab) = \operatorname{tr}(ba)$.

(2) $\operatorname{tr}(1) = 1$.

(3) $\operatorname{tr}(xg_n) = z\operatorname{tr}(x)$ *for* $x \in H_q(n)$

Remark (Skip)**.** Property (1) above implies that tr factors through

$$\bigcup_{n\ge 1} H_q(n) \to \bigcup_{n\ge 1} H_q(n)/[H_q(n), H_q(n)] = \text{Sym}$$

and some people also refer to the above map as the Jones-Oceanu trace, where you recover tr by specializing p_i to specific values.

Theorem 3.4 (Jones)**.** *Let $\beta \in B_n$. Define*

$$X_\beta(q,\lambda) = f(q,\lambda)\operatorname{tr}([\beta])|_{z=-\frac{1-q}{1-\lambda q}} \tag{11}$$

Then $P(\widehat{\beta})(a,q) = X_\beta\left(q^2, \dfrac{a^2}{q}\right)$.

Theorem 3.5 (Ocneanu)**.** *Let $x \in H_q(n)$*

$$\operatorname{tr}(x) = \sum_{\lambda\vdash n} \operatorname{Tr}_{S^\lambda(q)}(x) \prod_{(i,j)\in\lambda} \frac{q^i(1-q+z) - q^j z}{1 - q^{h_\lambda(i,j)}} \tag{12}$$

where the first row of λ has coordinates $(0,j)$ and the first column has coordinates $(i,0)$.

Proof. $H_q(n) = \bigoplus_{\lambda\vdash n} \text{End}(S^\lambda(q))$ is semisimple as q is generic. Any function $f : M_k(\mathbb{C}) \to \mathbb{C}$ satisfying $f(ab) = f(ba)$ is a scalar multiple of Tr. Ocneanu found these constants for us. ∎

3.1 Calculation of $P(T(m,n))$

Definition 3.6. *$T(m,n)$ is the closure of the braid $(\sigma_1 \dots \sigma_{n-1})^m$.*

Remark. $T(m,n)$ is a knot $\iff$ $(m,n)=1$ and $T(m,n)=T(n,m)$.

Let $\pi_\lambda : H_q(n) \to \mathrm{End}(S^\lambda)$.

Lemma 3.7. *Fix $\lambda \vdash n$ and define $e_i := \dfrac{1+\pi_\lambda(g_i)}{1+q}$. Then $e_i^2 = e_i$. Moreover let $\dim S^\lambda(q) = d$ and $\mathrm{rank}_{S^\lambda(q)} e_i = r$*[1]*. Then*

$$\pi_\lambda((g_1 \dots g_{n-1})^n) = q^{rn(n-1)/d}\mathrm{id}_{S^\lambda}$$

Proof. $e_i^2 = e_i$ is simple computation in $H_q(n)$.

Lemma 3.8. $\mathrm{FT}_n := (\sigma_1 \dots \sigma_{n-1})^n$ *is central in B_n.*

Because S^λ is irreducible it follows that

$$\pi_\lambda((g_1 \dots g_{n-1})^n) = c \cdot \mathrm{id}_{S^\lambda} \qquad c \in \mathbb{C}$$

By definition, $\pi_\lambda(g_i) = qe_i - (1-e_i)$. Because e_i is an idempotent it is diagonalizable with eigenvalues 1 and 0 and therefore in some basis of S^λ we have

$$\pi_\lambda(g_i) = qe_i - (1-e_i) = \begin{bmatrix} \overbrace{q \quad \ddots \quad q}^{\text{size } r} & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix}$$

Thus

$$\det(\pi_\lambda(g_i)) = \pm q^r \implies \det(\pi_\lambda((g_1 \dots g_{n-1})^n)) = q^{rn(n-1)} \implies c^d = q^{rn(n-1)} \implies c = w(q)q^{rn(n-1)/d}$$

where $w : \mathbb{C} \to \zeta_d$ where ζ_d is a $d-$th root of unity. w is continuous, $\mathbb{C}$ connected, and ζ_d is discrete and so w is constant. Note $g_1 \dots g_{n-1}|_{q=1} = (1\ n\ (n-1) \dots 2)$. $(n\ (n-1) \dots 1)^n = \mathrm{id}$ and so $1 = c|_{q=1} = w(1)$. ∎

Lemma 3.9. *The matrix $A_\lambda(q) = q^{-r(n-1)/d}\pi_\lambda(g_1 \dots g_{n-1})$ is conjugate to the matrix for the action of $(n\ (n-1) \dots 1)$ on S^λ.*

Proof. The previous lemma shows that $A_\lambda(q)^n - I = 0$. Therefore the minimal polynomial of A_λ has distinct roots and so A_λ is diagonalizable and so the conjugacy class is just determined by the eigenvalues and multiplicities of the eigenvalues. The same continuity and connectedness argument will show that these are constant and thus $A_\lambda(q)$ is conjugate to $A_\lambda(1) = (n\ (n-1) \dots 1)$. ∎

Corollary 3.10. *Suppose $(m,n) = 1$.*

$$\mathrm{Tr}(\pi_\lambda((g_1 \dots g_{n-1})^m)) = \begin{cases} (-1)^a q^{mr(n-1)/d} & \text{if } \lambda = H_{a,b} \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

where $H_{a,b}$ is the hook shape. [Draw hook with $a+1$ vertical boxes and $b+1$ horizontal boxes].

[1] Aprioi r depends on i but we will show it's independent later.

Proof. By the previous lemma we see that

$$\mathrm{Tr}(\pi_\lambda((g_1\dots g_{n-1})^m)) = q^{mr(n-1)/d}\mathrm{Tr}_{S^\lambda}((n\ (n-1)\dots 1)^m) = q^{mr(n-1)/d}\sum_i \lambda_i^m$$

where λ_i are the eigenvalues of $(n\ (n-1)\dots 1)$. As seen in previous lemma, all the λ_i are $n-$th roots of unity. As $(m,n)=1$ the map $\tau_m(w_n)=w_n^m$ where w_n a primitive $n-$th root of unity is in $\mathrm{Gal}(\mathbb{Q}(w_n)/\mathbb{Q})$ and note $\lambda_i^m=\tau_m(\lambda_i)$. As $\mathrm{char}((n\ (n-1)\dots 1))\in\mathbb{Q}[x]$ it follows that $\tau_m(\mathrm{char}_{(n\ (n-1)\dots 1)}(x)) = \mathrm{char}_{(n\ (n-1)\dots 1)}(x)$ and thus

$$\sum_i \lambda_i^m = \sum_i \tau_m(\lambda_i) = \tau_m\left(\sum_i \lambda_i\right) = \sum_i \lambda_i = \mathrm{Tr}_{S^\lambda}((n\ (n-1)\dots 1))$$

The result now follows from the Murnaghan-Nakayama rule. ■

Theorem 3 (Jones)
Suppose $(m,n)=1$. Then

$$P(T_{n,m})(a,q) = \frac{a^{m(n-1)}\langle 1\rangle_q}{\langle n\rangle_q}\sum_{b=0}^{n-1}(-1)^{n-1-b}\frac{q^{-m(2b-n+1)}}{\langle b\rangle_q!\,\langle n-1-b\rangle_q!}\prod_{\substack{j=b-n+1\\ j\neq 0}}\left(q^j a - q^{-j}a^{-1}\right) \tag{14}$$

where $\langle n\rangle = q^n - q^{-n}$.

Proof. Plug Eq. (13) into Eq. (12) and then plug that into Eq. (11). The only thing I haven't explicitly computed is $r=\mathrm{rank}_{S^\lambda(q)}(e_i)$ and $d=\dim S^\lambda$. First all the σ_i are conjugate to σ_1 in B_n as a result of the braid relations. So we only need to find $\mathrm{rank}\, e_1$. But $e_1\in H_q(2)$. Thus

$$\mathrm{rank}_{S^\lambda(q)}(e_1) = \mathrm{rank}_{\mathrm{Res}^{H_q(n)}_{H_q(2)}(S^\lambda(q))}(e_1)$$

There are 2 irreducibles for $H_q(2)$ and $\pi_{\square\square}(g_1)=q$ while $\pi_{\substack{\square\\ \square}}(g_1)=-1 \implies \pi_{\square\square}(e_1)=1$ and $\pi_{\substack{\square\\ \square}}(g_1)=0$ and so for $\lambda=H_{a,b}$

$$r=\mathrm{rank}_{S^{H_{a,b}}(q)}(e_1) = \text{mult of } S^{\square\square}(q) \text{ in } \mathrm{Res}^{H_q(n)}_{H_q(2)}(S^{H_{a,b}}(q)) \overset{\textit{branching}}{=\!=} \binom{a+b-1}{a}$$

and note we also have

$$d=\dim S^{H_{a,b}}(q) \overset{\textit{hook}}{=\!=} \binom{a+b}{a}$$

■

4 Cherednik Algebras and Torus Knots

Proposition 4.1 (Skip). *Let L be any representation of S_n, then*

$$\frac{1}{1-a}\mathrm{ch}(L;p_i=1-a^i) = \sum_{k=0}^{n-1}(-a)^k\dim_{\mathbb{C}}\mathrm{Hom}_{S_n}(\Lambda^k\mathfrak{h},L) \tag{15}$$

Proof. Applying definitions,

$$\sum_{k=0}^{n-1}(-a)^k \dim_{\mathbb{C}} \mathrm{Hom}_{S_n}(\Lambda^k\mathfrak{h}, L) = \sum_{k=0}^{n-1}(-a)^k \left\langle \Lambda^k\mathfrak{h}, L \right\rangle = \frac{1}{n!}\sum_{\sigma\in S_n}\sum_{k=0}^{n-1}(-a)^k \mathrm{Tr}_L(\sigma)\mathrm{Tr}_{\Lambda^k\mathfrak{h}}(\sigma)$$
$$= \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{Tr}_L(\sigma)\sum_{k=0}^{n-1}\mathrm{Tr}_{\Lambda^k\mathfrak{h}}(\sigma)(-a)^k$$

Writing out the first few terms, we see that

$$\sum_{k=0}^{n-1}\mathrm{Tr}_{\Lambda^k\mathfrak{h}}(\sigma)(-a)^k = 1 + \mathrm{Tr}_{\mathfrak{h}}(\sigma)(-a) + \ldots + \mathrm{Tr}_{\Lambda^{n-1}\mathfrak{h}}(\sigma)(-a)^{n-1}$$
$$= (-1)^{n-1}\,(\text{characteristic polynomial of } \sigma \text{ but coefficients reversed})$$
$$= (-1)^n q^n \mathrm{char}_\sigma\left(\frac{1}{q}\right) \overset{Eq.\ (2)}{=\!=} \det(I - q\sigma) \overset{Eq.\ (1)}{=\!=} \frac{1}{1-a}\prod_i (1-a^i)^{k_i(\sigma)}$$

Now apply the definition of $\mathrm{ch}(L)$. ■

Theorem 4 (GORS)
The graded Frobenius character of $L_{m/n}$ (after changing variables) coincides with the HOMFLY polynomial of the $(m,n)-$torus knot when $(m,n)=1$.

$$a^{(m-1)(n-1)}\frac{1}{1-a^2}\,\mathrm{gch}(L_{m/n})(q^2, p_i = (1-a^2)^i) = P(T_{n,m})(a,q)$$

Proof.

$$\frac{1}{1-a^2}\mathrm{gch}(L_{m/n})(q^2, p_i = (1-a^2)^i) \overset{Eq.\ (15)}{=\!=} \sum_i\sum_{k=0}^{n-1}(-a^2)^k \dim_{\mathbb{C}} \mathrm{Hom}_{S_n}(\Lambda^k\mathfrak{h}, (L_{m/n})_i)q^{2i}$$
$$\overset{check}{=\!=} \sum_i\sum_{k=0}^{n-1}(-a^2)^k \dim_{\mathbb{C}} \mathrm{Hom}_{S_n}(S^{a,n-1-a}, (L_{m/n})_i)q^{2i}$$
$$\overset{Eq.\ (10)}{=\!=} \text{ explicit function of } q \text{ and } a$$

One can then show using pro $q-$series manipulation to show that this is equal to Eq. (14). ■

Corollary 4.2 (rank-level duality)**.**

$$(L_{m/n})^{S_n} \cong (L_{n/m})^{S_m}$$

Fun Facts:

(a) $\mathrm{gr}\ L_{n+1/n} \cong \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]/\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]_+^{S_n}$

(b) $c_n(q,t) = \langle \nabla e_n, e_n\rangle = \mathscr{P}_{T(n,n+1)}(q,t,a=0)$